Kuwait University | MATH 102 | January 24, 2010
Dept. of Math.& Comp. Sci. | Final Exam | Duration: two hours

Calculators and mobile telephones are not allowed.

Answer the following questions.

1. Determine whether the statement is true or false. Justify your answer. **(1.5 pts. each)**

   (a) The function $f(x) = \begin{cases} -x & \text{if } -1 \le x < 0 \\ 3x+2 & \text{if } 0 \le x \le 1 \end{cases}$ is one-to-one on $[-1, 1]$.

   (b) The curve given by the parametric equations $x = 2\sin^2 t$, $y = 3\cos^2 t$ for $0 \le t \le \pi/2$ is a line segment from (0,3) to (2,0).

2. Find the solution set of the inequality $\log_{\frac{1}{2}}(x^2 - 3) > 0$. **(2 pts.)**

3. Evaluate the following limit, if it exists. **(3 pts.)**

$$\lim_{x\to\infty} \tan^{-1}\left(\frac{\sqrt{x^2+1}}{\ln x} + 1\right).$$

4. Determine whether the improper integral $\displaystyle\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx$ is convergent or divergent. If it converges, find its value. **(4 pts.)**

5. Evaluate the following integrals. **(4 pts. each)**

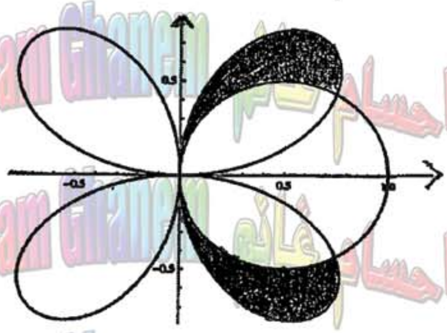
   (a) $\displaystyle\int \frac{\sinh x}{2\sinh^2 x + 3\cosh x}\,dx$ (b) $\displaystyle\int 2^x \cos x\,dx$

   (c) $\displaystyle\int \frac{1-\sqrt{1-x^2}}{x^2}\,dx$ (d) $\displaystyle\int \frac{x}{x^2-4x+8}\,dx$

6. Find an equation in rectangular coordinates of the tangent line to the graph of the polar equation $r = 4\cos^3\theta$ at the point corresponding to $\theta = \pi/4$. **(4 pts.)**

7. Find the area of the surface obtained by rotating the curve $y = \cosh x$, $0 \le x \le \ln\sqrt{2}$ about the x-axis. **(4 pts.)**

8. The figure below shows the graphs of the polar equations $r = \sin 2\theta$ and $r = \cos\theta$. Find the area of the shaded region. **(4 pts.)**



1. (a) **True** by the Horizontal Line Test.

   (b) **True** As $\frac{x}{2}+\frac{y}{3}=\sin^2 t+\cos^2 t=1$, the curve is on a line. The points which correspond to $t=0$ and $t=\frac{\pi}{2}$ are $(0,3)$ and $(2,0)$, respectively.

2. $\log_{\frac{1}{2}}(x^2-3)>0$ if and only if $\ln(x^2-3)<0$ if and only if $0<x^2-3<1$ if and only if $\sqrt{3}<|x|<2$, so the solution set is $(-2,-\sqrt{3})\cup(\sqrt{3},2)$.

3. $\lim_{x\to\infty}\tan^{-1}\left(\frac{\sqrt{x^2+1}}{\ln x}+1\right)=\frac{\pi}{2}$, since $\lim_{x\to\infty}\frac{\sqrt{x^2+1}}{\ln x}$ is of the form $\frac{\infty}{\infty}$ and by the L'Hospital's Rule $\lim_{x\to\infty}\frac{\sqrt{x^2+1}}{\ln x}=\lim_{x\to\infty}\frac{x^2}{\sqrt{x^2+1}}=\infty$ and $\lim_{u\to\infty}\tan^{-1}u=\frac{\pi}{2}$.

4. The improper integral is convergent and
$$\int_0^1\frac{\ln x}{\sqrt{x}}\,dx=\lim_{t\to 0^+}\int_0^1\frac{\ln x}{\sqrt{x}}\,dx=-2\lim_{t\to 0^+}\left(2+\sqrt{t}(\ln t-2)\right)=-4$$
We integrated by parts and for the limit we used the L'Hospital's Rule.

5. (a) Substituting $u=\cosh x$ and applying $\sinh^2 x=\cosh^2 x-1$ we get
$$\int\frac{\sinh x}{2\sinh^2 x+3\cosh x}\,dx=\int\frac{du}{2u^2+3u-2}=\int\frac{du}{(2u-1)(u+2)}=$$
$$=\frac{1}{5}\int\left(\frac{2}{2u-1}-\frac{1}{u+2}\right)du=\frac{1}{5}\ln\left|\frac{2\cosh x-1}{\cosh x+2}\right|+C$$

   (b) Integrating by parts twice, first with $u=2^x$ and $dv=\cos dx$, then with $u=2^x$ and $dv=\sin x dx$, we get
$$\int 2^x\cos x\,dx=2^x\sin x-\ln 2\int 2^x\sin x\,dx=2^x\sin x+\ln 2\cdot 2^x\cos x-(\ln 2)^2\int 2^x\cos x\,dx.$$
Hence
$$\int 2^x\cos x\,dx=\frac{2^x}{1+\ln^2 2}(\sin x+\ln 2\cdot\cos x)+C.$$

   (c) Using the trigonometric substitution $x=\sin\theta$ we have
$$\int\frac{1-\sqrt{1-x^2}}{x^2}\,dx=\int\frac{1-\cos\theta}{\sin^2\theta}\cos\theta d\theta=\int(\cot\theta\csc\theta-\csc^2\theta+1)d\theta=$$
$$=-\csc\theta+\cot\theta+\theta+C=-\frac{1}{x}+\frac{\sqrt{1-x^2}}{x}+\sin^{-1}x+C.$$

   (d) Completing the square
$$\int\frac{x}{x^2-4x+8}\,dx=\int\frac{x}{(x-2)^2+4}dx=\int\left(\frac{x-2}{(x-2)^2+4}+\frac{2}{(x-2)^2+4}\right)dx=$$
$$=\frac{1}{2}\ln(x^2-4x+8)+\tan^{-1}\left(\frac{x-2}{2}\right)+C.$$

6. The parametric equations are $x=4\cos^4\theta$ and $y=4\cos^3\theta\sin\theta$. When $\theta=\frac{\pi}{4}$, the slope is $m=\dfrac{dy/d\theta}{dx/d\theta}=\dfrac{\cos^2\theta-3\sin^2\theta}{-4\cos\theta\sin\theta}\Bigg|_{\theta=\frac{\pi}{4}}=\dfrac{1}{2}$ and the point is $P(1,1)$, hence an equation of the tangent line is $y-1=\frac{1}{2}(x-1)$.

7. The surface area is
$$\int_0^{\ln\sqrt{2}}2\pi\cosh x\sqrt{1+\sinh^2 x}\,dx=2\pi\int_0^{\ln\sqrt{2}}\cosh^2 x\,dx=\pi\int_0^{\ln\sqrt{2}}(1+\cosh 2x)\,dx=$$
$$=\pi\left(\ln\sqrt{2}+\frac{3}{8}\right).$$

8. From $\sin 2\theta=\cos\theta$ we get $\cos\theta(2\sin\theta-1)=0$, so $\theta=\pi/2$ and $\pi/6$. Hence the area is
$$A=2\int_{\pi/6}^{\pi/2}\frac{1}{2}\left[\sin^2 2\theta-\cos^2\theta\right]d\theta=\int_{\pi/6}^{\pi/2}\left(\frac{1-\cos 4\theta}{2}-\frac{1+\cos 2\theta}{2}\right)d\theta=\frac{3\sqrt{3}}{16}.$$